SOLVING QUADRATIC EQUATIONS BY THE DIAGONAL SUM METHOD.

(by Nghi H Nguyen)

A quadratic equation in one variable has as standard form: $ax^2 + bx + c = 0$. Solving it means finding the values of x that make the equation true.

Beyond the 4 known existing solving methods (quadratic formula, factoring, completing the square, and graphing), there is a new solving method, called Diagonal Sum Method, recently presented in book titled "New methods for solving quadratic equations and inequalities" (Trafford 2009). This new solving method can directly obtain the 2 real roots without factoring the equation.

It is a trial and error method, same as the factoring one, but it reduces the number of permutations in half by using a Rule of Signs for Real Roots. It is fast, convenient and is applicable whenever the equation is factorable. If this method fails to get the answer, then the equation is not factorable, and the quadratic formula must be used.

Innovative concept of the Diagonal Sum Method.

Direct finding 2 real roots, in the form of 2 fractions, knowing their sum (-b/a) and their product (c/a).

Rule of Signs for Real Roots.

If a and c have opposite signs, the real roots have opposite signs.
If a and c have same sign, the real roots have same sign:

a. If a and b have opposite signs, both real roots are positive.
b. If a and b have same sign, both roots are negative.

Examples of using the Rule of signs:

The equation $-6x^2 + 7x – 20 = 0$ has 2 real roots that have opposite signs
The equation $21x^2 + 50x + 24 = 0$ has 2 real roots, both negative.
The equation $12x^2 – 113x + 253 = 0$ has 2 real roots, both positive.

The diagonal sum of a set of 2 real roots.

Given a set of 2 real roots ($\frac{c_1}{a_1}$ & $\frac{c_2}{a_2}$). Their sum is equal to ($\frac{-b}{a}$).

The sum: $\frac{c_1}{a_1} + \frac{c_2}{a_2} = \frac{(c_1a_2 + c_2a_1)}{a_1.a_2} = \frac{-b}{a}$

The sum $(c_1a_2 + c_2a_1)$ is called the diagonal sum of a roots-set.

Rule for the Diagonal Sum.

The diagonal sum of a **true** roots-set must be equal to (-b). If it is equal to b, the set will be opposite in sign. If a is negative, the above rule is reversal in sign.

SOLVING QUADRATIC EQUATIONS IN VARIOUS CASES USING THE DIAGONAL SUM METHOD.

Depending on the values of the constants a and c, solving quadratic equations may be simple or complicated.

A. When a = 1. Solving the quadratic equation x^2 + bx + c = 0.

In this special case, the diagonal sum becomes the sum of the 2 real roots. Solving doesn't need factoring.

Example 1. Solve: x^2 - 21x – 72 = 0.

Solution. The Rule of Signs indicates the roots have opposite signs. Write the factors-sets of c = -72. They are: (-1, 72) (-2, 36) (-3, 24)…Stop here! The sum of the 2 roots in this set is 21 = -b. The 2 real roots are: -3 and 24.
Note. There are factors-sets in opposite sign (1, -72) (2, -36)…but they can be ignored since they are just opposite in sign. By convention, always put the negative sign in front of the first number.

Example 2. Solve: -x^2 – 26x + 56 = 0.

Solution. Roots have opposite signs. The constant a is negative. Write factor-sets of c = 56: (-1, 56) (-2, 28)…Stop here ! This sum is 26 = -b. According to the Rule for the Diagonal Sum when a is negative, the sum must be equal to (b). The true roots-set is opposite in sign to the set (-2, 28). The 2 real roots are: 2 and -28.

Example 3. Solve: x^2 + 27x + 50 = 0.

Solution. Both roots are negative. Write factors-sets of c = 50: (-1, -50) (-2, -25)…Stop here! This sum is -27 = -b. The 2 real roots are -2 and -25.

Example 4. Solve x^2 – 39x + 108 = 0.

Solution. Both roots are positive. Write factors-set of 108 = c.
They are: (1, 108) (2, 54) (3, 36)…Stop!. This sum is 3 + 36 = 39 = -b.
The 2 real roots are: 3 and 36.

B. When a and c are prime/small numbers.

The Diagonal Sum Method directly selects the probable roots-sets from values of the constants a and c and by applying the Rule of Signs.

a. When a and c are both prime numbers, the number of probable roots-sets is limited to one.

Example 5. Solve: 7x^2 + 90x – 13 = 0.

Solution. Roots have opposite signs.

Select the unique roots-set. The numerator contains factors-set of c = -13. The denominator contains factors-set of a = 7. The other roots-set can be ignored since 1 is not a real root.

Roots-set: $\left(\frac{-1}{7} , \frac{13}{1} \right)$ The diagonal sum: -1 + 91 = 90 = b

Since the diagonal is equal to b, the answers are opposite in sign: 1/7 and -13.

Example 6. Solve: 17x^2 + 324x + 19 = 0.

Solution. Both roots are negative. The constants a, c are both prime numbers.

Write down the unique roots-set: $\left(\frac{-1}{17} , \frac{-19}{1} \right)$. Its diagonal sum is: -323 -1 = -324 = -b.

The 2 real roots are: -1/17 and -19.

b. When a and c are small numbers, and contain themselves one or 2 factors, we may write down all probable roots-sets, then use mental math to calculate all diagonal sums and find the one that fits. Stop calculation when the diagonal sums match –b (or b).

Example 7. Solve: 7x^2 – 57x + 8 = 0.

Solution. Both roots are positive. The constant c = 8 has 2 factors-sets: (1, 8), (2, 4).

All probable roots-sets: $\left(\frac{1}{7} , \frac{8}{1} \right)$ $\left(\frac{2}{1} , \frac{4}{7} \right)$ $\left(\frac{2}{7} , \frac{4}{1} \right)$

Diagonal sums: (1 + 56 = 57 = -b) Answers: 1/7 and 8.

Example 8. Solve: 6x^2 – 19x – 11 = 0.

Solution. Roots have opposite signs. The constant a = 6 has 2 factors-sets: (1, 6), (2, 3)

Probable roots-sets: $\left(\frac{-1}{6} , \frac{11}{1} \right)$ $\left(\frac{-1}{2} , \frac{11}{3} \right)$ $\left(\frac{-1}{3} , \frac{11}{2} \right)$

Diagonal sums: (66 – 1 = 59) (22 - 3 = 19 = -b) Answers: -1/2 and 11/3

Note: There are opposite roots-sets (1/6 & -11/1)… but they can be ignored since they are just opposite in sign.

C. When a and c are large numbers and contain themselves many factors.

In these cases, to make sure that no roots-sets may be omitted, it is advised to write all the combinations on an all-options-line. On this line, the numerator contains all factors-sets of c. The denominator contains all factors-sets of a.

Example 9. Solve: $8x^2 + 13x - 6 = 0$.

Solution. Roots have opposite signs. Write the all-options-line:

Factors-sets of c = -6: (-1, 6) (-2, 3)
Factors-sets of a = 8: (1, 8) (2, 4)

Now, you can use mental math, or a calculator, to calculate all diagonal sums and find the one that fits.
It is the option (-2 , 3) that gives as diagonal sum: -16 + 3 = -13 = -b.
1 8
The 2 real roots are: -2 and 3/8.

Example 10. Solve: $45x^2 - 74x - 55 = 0$.

Solution. Roots have opposite signs. Write the all-options-line.

Numerator. Factors-sets of c = -55: (-1, 55) (-5, 11)
Denominator. Factors-sets of a = 45: (1, 45) (3, 15) (5, 9)
You can use mental math, or a calculator, to calculate all diagonal sums and find the one that fits. You may also proceed by elimination. First, eliminate the options (-1, 55)
(1, 45) (3, 15)
because they give large diagonal sums, compared to b = -74.
The remainder option (-5 , 11), gives unique roots-set (-5 , 11)
(5, 9) 9 5
Its diagonal sum is -25 + 99 = 74 = -b. The 2 real roots are: -5/9 and 11/5.

Example 11. Solve: $12x^2 - 272x + 45 = 0$.

Solution. Both roots are positive. Write all-options-line.
Factors-sets of c = 45: (1, 45) (3, 15) (5, 9)
Factors-sets of a = 12; (1, 12) (2, 6) (3, 4)
First, eliminate options linked to (1, 12) and (3, 4) since they give odd-number-diagonal-sums while b is an even number. Second, look for a large-number diagonal-sum (-272). The fitted option should be (1, 45), that gives the 2 real roots: 1/6 and 45/2.
(2, 6)

Example 12. Solve: $45x^2 - 172x + 36 = 0$.

Solution. Both roots are positive. Write all-options-line:

(1, 36) (2, 18) (3, 12) (6, 6)
(1, 45) (3, 15) (5, 9)
First, eliminate options (1, 36) (3, 12) since they give odd-number-diagonal-sums. Then, eliminate options (6, 6)/(3, 15) because they give the diagonal-sums that are multiple of 3. This would make the given equation to be simplified by 3, and that is not the wanted solution. The remainder options are (2, 18)/(1, 45) and (2, 18)/(5, 9). The second option gives 2 probable roots-sets: $\left(\frac{2}{5}, \frac{18}{9}\right)$ and $\left(\frac{2}{9}, \frac{18}{5}\right)$.
The second set gives as diagonal sum 172 = b. The 2 real roots that are: 2/9 and 18/5.

Example 13. Solve: 12x^2 + 5x – 72 = 0.

Solution. Roots have opposite signs. Write all-options-line.
Numerator: (-1, 72) (-2, 36) (-3, 24) (-4, 18) (-6, 12) (-8, 9)
Denominator: (1, 12) (2, 6) (3, 4)
First, eliminate options with (-2, 36) (-4, 18) (-6, 12) /(2, 6) since they give even-number-diagonal-sums (b is odd). Then, eliminate options (-1, 72) (-3, 24) /(1, 12) because they give large-number-diagonal-sums (b = 5). The remainder option is (-8, 9)/(3, 4). This gives two probable roots-sets $\left(\frac{-8}{3}, \frac{9}{4}\right)$ and $\left(\frac{-8}{4}, \frac{9}{3}\right)$.
The first set gives as diagonal sum (27 – 32 = -5 = -b). The 2 real roots are: -8/3 and 9/4.

Example 14. Solve: 24x^2 – 59x + 36 = 0.

Solution. Both roots are negative. Write the all-options-line.
Numerator: (-1, -36) (-2, -18) (-3, -12) (-4, -9) (-6, -6)
Denominator: (1, 24) (2, 12) (3, 8) (4, 6)
First, eliminate options (-2, -18) (-6, -6)/ (2, 12) (4, 6) because they give even-number-diagonal-sums (b is odd). Then, eliminate options (-1, -36) (-2, -18) /(1, 24) since they give too large diagonal-sums (b = -59). The remainder option is the set $\frac{(-4, -9)}{(3, 8)}$ that gives the two real roots: -4/3 and -9/8.

Comments.

You may ask a good question: "Why to do these complicated operations while I can quickly get the answer by using the quadratic formula with a calculator?"
Here is also a good answer: Performing these operations helps fulfill the goal of learning math that is to improve logical thinking and deductive reasoning. Imagine a situation in which you have to solve these complex equations by the quadratic formula without a calculator, during some tests/exams for examples. It would be a boring hard work and it won't even guarantee a true answer.

(This article was written for LTAHCC by Nghi H. Nguyen, the co-author of the new Diagonal Sum Method)